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Book review

Rescue the Gardiner book!

Crispin Gardiner, Physics Department, University of Otago, Dunedin, NEW ZEALAND
ISBN 3-540-20882-8, 2004 (xvii + 415pp., $84.82).

Reviewed by Raoul P.P.P. Grasman, Eric-Jan Wagenmakers

The book author, professor Crispin Gardiner, is currently honorary fellow at the physics department of Victoria University of Wellington, New Zealand. He received his Ph.D. from Oxford University, England, and has since been working at various institutes, including the University of Waikato, Hamilton, New Zealand. Over the years, he has made many contributions to the study of Bose–Einstein condensation, quantum optics, and stochastic methods in physics and chemistry. His website can be found at http://www.physics.otago.ac.nz/people/gardiner/index.html.

The reviewers, Raoul Grasman and Eric-Jan Wagenmakers, both received their Ph.D. at the University of Amsterdam. Their common research interests include sequential sampling models for choice response times, catastrophe theory, and the analysis of (nonlinear) dynamical systems with random components.

Suppose your house is on fire, and you only have time to rescue a single item from the blazing inferno. What item would you choose to save? A Dutch mathematics professor—happily married, with children—instantly replied: "I would rescue the Gardiner book!". This anecdote illustrates at least two points. The first point is that the book by Crispin Gardiner (2004), "Handbook of Stochastic Methods" is a classic text on stochastic differential equations (SDEs). The second point is that the mathematics professor was probably unaware that the out-of-print 1985 edition has been followed up by a 1997 edition, and, most recently, a 2004 edition.

Gardiner's *Handbook* is dedicated to mathematical models for time varying systems that involve stochastic components. Although the book is geared towards applications in physics and chemistry, the book has become well known in fields such as mathematical finance. Stochastic processes have a long history in physics, chemistry and biology, where dynamical systems often involve stochastic components. In cognitive psychology, dynamic stochastic processes have found application in the modeling of simple decision processes (Ratcliff, 1978; Smith & Ratcliff, 2004), decision field theory (Busemeyer & Townsend, 1992), and the multi-attribute dynamic decision model (Diederich, 1997). Smith (2000) provides an extensive introduction into the subject.

The book provides a detailed and systematic development of techniques of mathematical analysis of stochastic processes, bringing together a wide range of exact, approximate, and numerical methods for solving SDEs. Compared to other volumes on stochastic processes and SDEs (e.g. Arnold, 2003; Doob, 1953; Feller, 1971; Klebaner, 1998; Øksendal, 2000), the book is written in a informal, narrative and nonmathematical style. Nevertheless, the book is not written for the mathematically timid.

The first five chapters are certainly accessible for advanced undergraduates of physics and mathematics. These chapters should be equally suitable for graduate students in biology, econometrics and mathematical psychology, with a similar level of mathematical background. An especially useful characteristic of the book is that the material is presented in chunks that can to some extent be studied independently—in contrast to most other books that we know of, a full read of the chapters preceding the chapter of interest is not necessary. One can even study an individual section without the need to study the preceding chapters, although a basic grasp of the material presented in the first four chapters certainly helps. A further characteristic of the book that is worth mentioning is the attention devoted to stochastic nonlinear dynamics, including bistability and multistability: Chapter 9 is entirely devoted to this topic.

The book is organized as follows: the first four chapters develop key concepts in stochastic processes and outline two different approaches to their analysis. Key concepts include Brownian motion and birth–death processes, the diffusion equation, Langevin's equation, the Markov assumption, the Chapman–Kolmogorov equation, the differential form of the Chapman–Kolmogorov equation which gives rise to both the Master equation and the Fokker–Planck equations, and the Itô notion of a SDE and associated calculus. The subsequent chapters deal with techniques for solving Fokker–Planck equations (Chapters

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5 and 6) and the Master equation (Chapters 7 and 8), techniques for the analysis of bistable and metastable systems (Chapter 9), and techniques for simulation—or numerical solution—of SDEs (Chapter 10).

Before giving a more in depth overview of the subjects treated in the book, we briefly discuss the concept of a random dynamical system.

1. Random dynamical systems

The concept of a dynamical system whose evolution involves random sample paths originated in biology: The eminent British biologist Robert Brown is usually credited for being the first to describe the typical motion of particles immersed in a solution observed under a microscope—a motion that now carries his name.[1] Brown had observed the typical motion of pollen, and excluded the presence of life as an explanation for such motion by replicating the result with anorganic particles. Exactly 100 years ago, Albert Einstein was the first to give a physical derivation of an exact mathematical description of Brownian motion. However, the mathematical ideas involved were conceived earlier by Bachelier (1900; cited in Courtault et al., 2000) in his analysis of speculation-based trade. Inspired by the molecular-kinetic theory of heat, Einstein figured that the motion of Brown's pollen was due to the random impacts of the molecules of the solution. Taken together, these very many random impacts would displace the particle by a random distance. In his derivation, Einstein assumed the existence of a continuous differentiable distribution law that governed these random displacements. He then reasoned about the number of particles $f(x, t+\tau)$ that would exist at time $t+\tau$ at a certain location x given the distribution of particles across space at time t. Through approximations, Einstein was able to deduce that the change in the number of particles over time was governed by the equation

$$\frac{\partial f}{\partial t} = D\frac{\partial^2 f}{\partial x^2}. \tag{1}$$

This equation is independent of the particular distribution law that governs the random displacements. Only the variance of the displacements shows up in the constant D. This equation is the well-known *diffusion equation*,, and has the solution

$$f(x,t) = \frac{n}{\sqrt{2\pi(2Dt)}} e^{-x^2/4Dt}.$$

One should recognize the scope of this result: although the exact path of a single particle is still unknown (one should not expect it to be known so long as the random impacts are unknown), the entire distribution of the particles across space is known for all times t! In essence, the program for dealing with stochastic dynamical systems as instigated by Einstein, was to find differential equations such as (1), that govern the probability density functions and then use conventional techniques to solve these equations. Eq. (1) is an example of a Fokker–Planck equation—a class of equations that play a central role in this program.

An entirely different line of attack of Brownian motion was proposed by Langevin, some time after Einstein's solution. Langevin reasoned that the movements of a particle immersed in a fluid would be governed by Newton's second law

$$F = m\frac{d^2x}{dt^2},$$

and that the force F on the particle consisted of the sum of two parts: the first part is Stokes drag force $-6\pi\mu a\vec{v}$, where μ is the viscosity of the fluid, a is the diameter of the particle (modeled as a sphere) and $\vec{v}$ is the velocity of the particle, and the second part is a random impact η on the particle, which represents the aggregate of a great many smaller impacts. The particle's movements would therefore be governed by the equation

$$m\frac{d^2x}{dt^2} = -6\pi\mu a v + \eta, \tag{2}$$

where $v = \|\vec{v}\|$ is the speed. Because η is unknown, this equation cannot be solved for x. However, an equation for its statistical characterization can be derived if the equation is multiplied by x and rewritten as follows:

$$\frac{m}{2}\frac{d^2x}{dt^2} + 3\pi\mu a\frac{d(x^2)}{dt} = mv^2 + \eta x.$$

Langevin argued that the expected value of the second term vanishes because the rapidly fluctuating random displacements η are independent of x. Therefore, if we take expectations on both sides, and use the Equipartition Theorem, (Gershenfeld, 1999) which says that the average of $mv^2/2$ equals $kT/2$, where k is Boltzmann's constant and T is the absolute temperature, we find the equation for the variance of x

$$\frac{d^2\sigma^2}{dt^2} + \frac{3\pi\mu a}{m}\frac{d\sigma^2}{dt} = 2kT/m.$$

The solution of this equation is given by $\sigma^2(t) = A\exp(-6\pi\mu a t/m) + [kT/3\pi\mu a]t$, where A is an arbitrary constant. Langevin determined that the first term rapidly decreases to zero within a minute time interval, and so, for any practical purpose, the variance of the particle's path amounts to

$$\sigma^2(t) = [kT/3\pi\mu a]t.$$

This agrees with Einstein's result if we set $D = [kT/3\pi\mu a]$. But its derivation was, at least according to Langevin, "infinitely more simple." Indeed the method of Langevin is much more direct than the method of Einstein, but its mathematical foundation turned out to be much more delicate than Langevin anticipated. A key step in

[1] In fact, the Dutch-born British physiologist Jan IngenHousz observed and described Brownian motion some 37 years earlier, in 1785.

Langevin's derivation is the assumption that the random displacements η and x are independent. This is also a key step in Einstein's derivation, but in the form of Langevin it was not until Itô developed his stochastic calculus (Itô, 1944) that this step was made precise. The program of handling stochastic processes using Langevin's approach is to consider equations such as (2), and to derive the characteristics of processes governed by it directly from the equation.

2. Systematic treatment of stochastic processes

A systematic approach to the analysis of stochastic processes along Einstein's reasoning can be developed from the *Chapman–Kolmogorov equation*. The development of a systematic approach along Langevin's reasoning requires the introduction of a notion of *stochastic integral*. We now discuss these approaches in turn.

2.1. The Chapman–Kolmogorov, Fokker–Planck and Master equations

In general, a stochastic process X_t can be described by all joint probability densities of the form

$$p(x_{t_1}, x_{t_2}, \ldots, x_{t_m}) \equiv p(x_1, t_1; x_2, t_2; \ldots; x_m, t_m),$$

for all possible choices of $t_1, t_2, \ldots, t_m$, and m. Obviously, this is a very general description and not much more can be said. If one restricts attention to special classes of processes, defined by specific forms of these joint densities, more progress can be made. Any joint density $p(x_1, t_1; x_2, t_2, \ldots)$ can be expressed in terms of conditional densities as the product $p(x_1, t_1|x_2, t_2, \ldots)p(x_2, t_2; \ldots)$. Assume that $t_1 > t_2 > \cdots$. One of the simplest forms for the joint densities is the form that can be expressed in conditional densities that are completely determined by the most recent state of the process, i.e.,

$$p(x_1, t_1|x_2, t_2, \ldots)p(x_2, t_2; \ldots) = p(x_1, t_1|x_2, t_2)p(x_2, t_2; \ldots),$$

for all possible choices of $t_1, t_2, \ldots$. This is called the *Markov assumption*, and a process satisfying the assumption is called a *Markov process*. If this equation holds for a process X_t, the conditional probability density $p(x_1, t_1; x_3, t_3)$ can be written as follows:

$$p(x_1, t_1 | x_3, t_3) = \int dx_2\, p(x_1, t_1|x_2, t_3)p(x_2, t_2|x_3, t_3), \quad (3)$$

which has the interpretation that the likelihood for the process to go from x_3 to x_1 is the sum of the likelihoods of all possible paths for X_t to get there. Eq. (3) is called the *Chapman–Kolmogorov equation*, and is fundamental to all Markov processes. It is the starting point for a general theory of stochastic processes in terms of the time evolution of the probability density of X_t in line with Einstein's description of Brownian motion. To this end, the Chapman–Komogorov equation can be put into a differential form, provided certain regularity conditions are met. The regularity conditions concern a probabilistic conception of continuity of the sample path of X_t, which amount to continuity in an almost sure sense (i.e., allowing only discontinuities on a zero-measure set of time points). The resulting differential Chapman–Kolmogorov equation entails the sum of two parts: the *Master equation* and the *Fokker–Planck equation*. The Master equation describes jumps of the process—that is, it describes the occurrence of discontinuities. The Fokker–Planck equation, which itself is composed of two parts, describes the continuous part of the process—the first part of the Fokker–Planck equation corresponds mathematically to a *drift process*, the second to a *diffusion process*. Each of these components may or may not be present in a stochastic process. An example of a combination of these processes is displayed in Fig. 1.

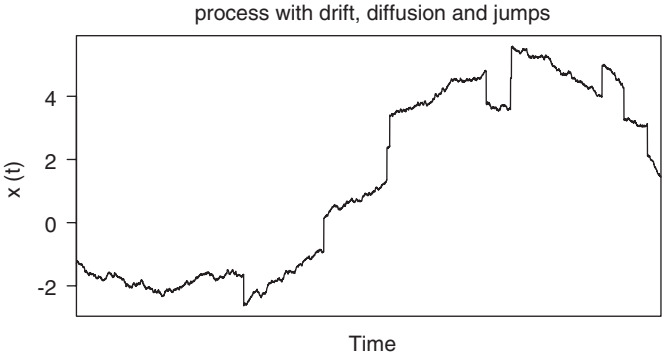

Fig. 1. Example of general stochastic process described by both a Fokker–Planck part (drift and diffusion) and a Master equation part (jumps).

2.2. Langevin approach: stochastic differential equation

The conceptual simplicity of the approach of Langevin to Brownian motion is complemented by the mathematical complexity in that, while Brownian motion as described by Einstein has sample paths that are continuous, these sample paths are *nowhere differentiable*. Yet it must be expected that the Langevin equation (2) is integrable. That is, we must expect $\int^t \eta(s)\,ds$ to exist. Now, if $\int^t \eta(s)\,ds$ is assumed to be a continuous function of t, it must be a continuous Markov process and hence can be described by a Fokker–Planck equation. However, it turns out that the corresponding Fokker–Planck equation is in fact Eq. (1), and hence, "$\int^t \eta(s)\,ds$" is a non-differentiable Brownian motion. While, therefore, Langevin's equation (2) cannot be interpreted consistently as a differential equation, a consistent interpretation of its integral form

$$x(t) = \int^t \beta x(s)\,ds + \int^t \eta(s)\,ds$$

can be given if the second integral is interpreted as a kind of Stieltjes–Lebesgue integral. Here, we enter the realm of stochastic integration: Let G be a suitably behaved function of time, let the interval $[t_0, t]$ be partitioned by the ordered sequence $t_1, t_2, \ldots, t_n$, and let $\tau_1, \tau_2, \ldots, \tau_n$ be intermediate points such that $t_{i-1} \leqslant \tau_i \leqslant t_i$ for $i = 1, \ldots, n$. Then the stochastic integral $\int_{t_0}^{t} G(s)\, dB(s)$ is defined as the $n \to \infty$ limit of the partial sums

$$S_n = \sum_{i=1}^{n} G(\tau_i)[B(t_i) - B(t_{i-1})],$$

where $B(t)$ is a Brownian motion process. With this definition, a SDE as envisioned in the Langevin equation can be written as follows:

$$dx(t) = a[x(t), t)\, dt + b[x(t), t]\, B(t), \tag{4}$$

as long as we interpret this equation as the corresponding integral equation $x(t) = \int_{t_0}^{t} a[x(s), s]\, ds + \int_{t_0}^{t} b[x(s), s]\, dB(s)$, where the latter term denotes the stochastic integral as defined above. Many equations encountered in practical physical modeling can be put into this form (possibly by conversion to a multivariate SDE).

The above definition of stochastic integral is ambiguous with respect to the choice of the τ_i's. Whereas in conventional calculus it can be shown that the choice is immaterial, and all possible choices yield the same result, this is not so with stochastic integrals. Depending on the choice of the τ_i's one obtains different definitions of stochastic integral, each with different values, and each obeying a different set of calculus rules. The mathematically most convenient choice—which leads to the concept of *non-anticipating functions* and is adopted in the Itô calculus—is the choice $\tau_i = t_{i-1}$. Another choice is $\tau_i = (t_{i-1} + t_i)/2$, and this is the choice that is adopted in the Stratonovich interpretation of stochastic integral. The Stratonovich interpretation is considered physically most satisfactory (see the discussion by van Kampen, 1981a, 1981b, 2001). Non-anticipating functions can be considered as a precise specification of what Langevin indicated by the assumed independence of x and η on which his derivation relies—in the Stratonovich choice these would in fact be correlated.

3. The Fokker–Planck equation

For the process described by the above differential equation (4), the Itô calculus can be put to use to derive a corresponding Fokker–Planck equation that describes the time evolution of the conditional probabilities of the process. It is given by

$$\begin{aligned}\partial_t p(x, t|x_0, t_0) = &-\partial_x[a(x, t)p(x, t|x_0, t_0)] \\ &+ \tfrac{1}{2}\partial_x^2[b(x, t)^2 p(x, t|x_0, t_0)].\end{aligned} \tag{5}$$

This is a key result for diffusion processes, and it allows one to switch back and forth between Einstein's approach and Langevin's approach, thus making available the analytic techniques developed in both worlds. Indeed, some results for diffusion processes are obtained much more directly from the Fokker–Planck equation than from the corresponding SDE; as evidenced by Langevin's derivation of Einstein's result, the reverse can also be true.

In sum, the central equations in the study of diffusion processes are Eqs. (4) and (5), which serve as the starting point for analysis of specific instances of these equations. For jump processes or discrete-event processes, the Master equation is the starting point.

With these equations in position, Gardiner's book continues in Chapter 5 with a treatment of systematic methods for finding answers to questions such as "What is the distribution of the process at time t (given an initial condition)?", "How long, on average, does it take for a process to leave a specific region—i.e., what is its mean exit time?", "To what stationary distribution does the distribution evolve as $t \to \infty$?", "What are the moments of the process at any time?", and "What is the distribution of the process's exit times?".

In order to answer such questions it is necessary to impose initial and/or boundary conditions. The first part of Chapter 5 provides the nontrivial derivation of appropriate boundary conditions. For most of the above questions exact answers can be obtained for certain forms of the Fokker–Planck equation of single variable processes. For multi-variable processes the answers are less explicit and certainly more complicated. For more general forms of these equations that fall outside the scope of these particular forms, exact answers cannot be given and one has to resort to approximation methods.

Prominent systematic approximation methods are treated in Chapter 6. The two main classes of systematic methods of approximation for the Fokker–Planck equation are *small noise perturbations* and *adiabatic elimination*. Both yield asymptotic results in the large t limit.

Small noise expansions are perturbation expansions that apply to situations in which the noise is very small—e.g., arising from thermal fluctuations. The method constructs an infinite set of ordinary or SDEs that can be solved sequentially, and whose solutions form a basis on which the solution can be expanded. Usually, only the first two equations are necessary. The book points out how such perturbation expansions differ as they are derived for either SDEs or for Fokker–Planck equations, discusses their validity, indicates when they go awry, and explains how they can be used to predict moments of the processes they describe.

A large section of Chapter 6 is devoted to adiabatic elimination—a theory to whose systematic treatment the book's author has made a significant contribution in his 1984 paper (Gardiner, 1984). Adiabatic elimination refers to the elimination of fast varying components of a process, in a situation of coupled processes that each fluctuate on very different time scales. The faster varying components tend to contribute very little to the overall stochastic behavior and are therefore eliminated completely. This is exactly as in Langevin's original reasoning, but put on a more general and sound mathematical foundation.

4. The master equation

The prototypical example of a process described with a Master equation is the population dynamics in a predator–prey system. Except for specific situations (e.g., the time homogenous Master equation) not many systematic methods are available for the Master equation. The fundamental result is that any diffusion process that can be modeled by a Fokker–Planck equation can be arbitrarily closely approximated by a process described by a Master equation. For example, a diffusion process can be arbitrarily closely mimicked by a random walk. In certain cases, the opposite is true as well, and if so, a macroscopic law of dynamics emerges from the detailed lower level dynamics as the system size is increased. The circumstance under which this can be achieved, and methods to do so, are largely the subject of Chapter 7. In particular, the chapter provides an elaborate treatment of the *system size expansion* technique as well as the *Poisson representation* technique for transforming Master equations into Fokker–Planck equations. The Poisson representation technique is an invention of the book's author (Gardiner & Chanwedi, 1977), who considers it to be his major contribution to the analysis of stochastic dynamical systems.

5. Simulation of SDEs

The book concludes with a chapter on numerical solution of SDEs, or their stochastic simulation. The intuitively most obvious method is Euler's method which is probably also the most widely used. This method is discussed along with other algorithms that show improved convergence (i.e., the simulated process stays within a close neighborhood of the modeled process), with the numerical stability of these algorithms (which concerns the propagation of the errors made by the algorithm with increasing time intervals for which the equation is solved), and with consistency (i.e., the simulated process converges to the modeled process as the time step interval decreases to zero). Furthermore, Chapter 10 provides algorithms for simulating multivariate SDEs and discusses algorithms for Stochastic Partial Differential Equations.

6. Conclusion

Gardiner's *Handbook of Stochastic Methods* is a very useful reference for anyone who wants to deepen their understanding of stochastic processes, and wants to gain confidence in applying the available techniques for stochastic processes in cognitive modeling. Even though the topics and examples are directed towards physics and chemistry, and the treatment goes far beyond the level of modeling detail currently achievable in cognitive psychology, the book contains a wealth of information that could be put to use directly by mathematical psychologists. Every topic in the book is firmly grounded in both an advanced mathematical treatment, as well as in clear intuitive explanations and examples. The resulting informal style of discourse and emphasis on intuition makes the book ideal for self-study, and promotes a rapid advance in mastering the subject.

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